ON THE INFLUENCE OF TURBULENT MOTIONS ON NON-RADIAL OSCILLATIONS

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The effect of turbulent motions on oscillations is studied, considering only the coupling between turbulent and oscillatory velocities. In this case, the turbulence affects the oscillations through the Reynolds stresses in the momentum equation for the pulsations. A simple model of turbulence is adopted to evaluate these Reynolds stresses and the perturbed eigenfrequencies are expressed as a function of certain averages of the turbulent velocities.

INTRODUCTION

The study of time-dependent convection (Unno 1967; Gabriel et al. 1975; Gough 1977; Deupree 1977; Xiong 1977; Keeley 1977; Goldreich and Keeley 1977; Baker and Gough 1979; Saio 1980; Gonczi and Osaki 1980) is an important subject for the understanding of pulsating stars with convection zones.

Several of the above papers study the generalization of the mixing length theory of convection to pulsating stars and the influence of pulsation on the velocity and temperature fluctuations must be considered.

The aim of this paper is far more modest; we retain the influence of the oscillations on the turbulent velocities only, and neglect it on the turbulent values of the pressure, density and temperature. Therefore, the effect of the turbulent motions on the oscillations manifests itself solely through the mean Reynolds stresses in the momentum equation for the pulsations. The definition of mean quantity is based on time averages: it is assumed that the oscillations remain coherent for a time much longer than the life-time of the turbulent eddies. To calculate the mean Reynold stresses we adopt a very simple model of

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turbulence; namely, we assume that the turbulent velocities remain unchanged for a turnover time and then suddenly change to an uncorrelated velocity field. The perturbed equation for the oscillations is solved by a variational principle and the perturbed eigenfrequencies, ω', are expressed as a function of ω_0, the turbulent eddies' lifetime and certain averages of the turbulent velocities.

THEORY

a) The Momentum Equation

The equation for conservation of momentum can be written (we neglect rotation):

$$D\underset{\sim}{u}/Dt = \nabla\psi - \rho^{-1}\nabla p, \tag{1}$$

where $\underset{\sim}{u}$, ψ, ρ, and p are, respectively, the fluid velocity, the gravitational potential, the density and the pressure. The convective rate of change operator, D/Dt, is defined by

$$D/Dt = \partial/\partial t + \underset{\sim}{u} \cdot \nabla. \tag{2}$$

For $\underset{\sim}{u}$, p, ρ, and ψ we can write

$$\underset{\sim}{u} = \overline{\underset{\sim}{u}}\, e^{i\omega t} + \underset{\sim}{u}', \tag{3a}$$

$$p = \overline{p}\, e^{i\omega t} + p_0 + p_0', \tag{3b}$$

$$\rho = \overline{\rho}\, e^{i\omega t} + \rho_0 + \rho_0', \tag{3c}$$

$$\psi = \psi_0. \tag{3d}$$

In Eqs. (3), the quantities with a bar are oscillating quantities; p_0, ρ_0 and ψ_0 are the pressure, density and gravitational potential in the absence of turbulence and oscillations; $\underset{\sim}{u}'$, p_0', ρ_0' are the turbulent velocity, pressure and density. It is clear from Eq. (3d) that we neglect the effect of the oscillation and turbulent motions on the gravitational potential. The subscript '0' in p_0' and ρ_0' indicates that we also neglect the effect of the oscillation on the turbulent values of the pressure and density, i.e., we assume, for example, that $p'(\overline{\underset{\sim}{u}} \neq 0) = p'(\overline{\underset{\sim}{u}} = 0) = p_0'$. We stress, on the other hand, that we do not neglect the effect of the oscillation on $\underset{\sim}{u}'$. In fact the derivation of the equation relating $\underset{\sim}{u}'$ and $\underset{\sim}{u}_0'$ is one of the main objects of this paper (the subscript '0' designates, here and elsewhere, quantities in the absence of oscillations).

We expand $1/\rho$ as follows (see Eq. (3c)):

$$\frac{1}{\rho} = \frac{1}{\rho_0}\left(1 - \frac{\overline{\rho} e^{i\omega t}}{\rho_0} - \frac{\rho_0'}{\rho_0}\right). \tag{4}$$

With the help of the unperturbed equation (Eq. (5b) below) and of Eqs. (3) and (4) it is straightforward to show that Eq. (1) can be written as follows:

$$i\omega\bar{\underset{\sim}{u}}e^{i\omega t} + \partial\underset{\sim}{u}'/\partial t + e^{i\omega t}\,\bar{\underset{\sim}{u}} \cdot \nabla\underset{\sim}{u}' + e^{i\omega t}\,\underset{\sim}{u}' \cdot \nabla\bar{\underset{\sim}{u}}$$

$$+ \underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}' - \langle\underset{\sim}{u}'_0 \cdot \nabla\underset{\sim}{u}'_0\rangle_E = - e^{i\omega t}\,\rho_0^{-1}\,\nabla\bar{p} - \rho_0^{-1}\,\nabla p'_0$$

$$+ e^{i\omega t}\,\bar{\rho}\rho_0^{-2}\,\nabla p_0 + e^{i\omega t}\,\bar{\rho}\,\rho_0^{-2}\,\nabla p'_0$$

$$+ \rho'_0\rho_0^{-2}\,\nabla p_0 + e^{i\omega t}\,\rho'_0\rho_0^{-2}\,\nabla\bar{p} + \rho'_0\rho_0^{-2}\,\nabla p'_0 - \rho_0^{-2}\,\langle\rho'_0\,\nabla p'_0\rangle_E \qquad (5a)$$

$$-\langle\underset{\sim}{u}'_0 \cdot \nabla\underset{\sim}{u}'_0\rangle_E - \frac{1}{\rho_0}\,\nabla p_0 + \rho_0^{-2}\,\langle\rho'_0\nabla p'_0\rangle_E + \nabla\phi_0 = 0. \qquad (5b)$$

In Eqs. (5) the bracket denotes an ensemble average and in Eq. (5a) we have neglected terms quadratic in the oscillations. We assume now that the oscillations remain coherent for a time much longer than the life-time of the turbulent eddies. We multiply Eq. (5a) by $e^{-i\omega t}/T$ and integrate in time from zero to T. In the limit $T \to \infty$ we obtain

$$i\omega\bar{\underset{\sim}{u}} + e^{-i\omega t}\,\langle\underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\rangle = - \frac{1}{\rho_0}\,\nabla\bar{p} + \frac{\bar{\rho}}{\rho_0^2}\,\nabla p_0, \qquad (6)$$

with

$$\langle\underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\rangle = e^{i\omega t}\,\lim_{T\to\infty}\frac{1}{T}\int_0^T \underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\,e^{-i\omega t}\,dt. \qquad (7)$$

The justification of Eq. (6) is straightforward, since it is clear that an integral of the form $\frac{1}{T}\int_0^T L'dt$ where L' is linear in the turbulent quantities, vanishes for large T's. It is also clear that the same holds true for $\frac{1}{T}\int_0^T \frac{\rho'_0\nabla p'_0}{\rho_0^2}\,e^{-i\omega t}\,dt$. The quantity $\langle\underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\rangle$, on the other hand, differs from zero; this is due to the influence of the oscillation on the turbulent motions (see Eq. (9) below).

Equation (6) plays the role of the mean equation and we proceed now to evaluate $\langle\underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\rangle$. We multiply Eq. (6) by $e^{i\omega t}$ and subtract it from Eq. (5a). It is readily found that the fluctuating momentum equation is given by

$$\partial\underset{\sim}{u}'/\partial t + e^{i\omega t}\,\bar{\underset{\sim}{u}}\nabla\underset{\sim}{u}' + e^{i\omega t}\underset{\sim}{u}' \cdot \nabla\bar{\underset{\sim}{u}}$$

$$+ \underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}' - \langle\underset{\sim}{u}'_0 \cdot \nabla\underset{\sim}{u}'_0\rangle_E - \langle\underset{\sim}{u}' \cdot \nabla\underset{\sim}{u}'\rangle$$

$$= -\rho_0^{-1}\nabla p_0' + e^{i\omega t}\bar{\rho}\rho_0^{-2}\nabla p_0' + \rho_0'\rho_0^{-2}\nabla p_0$$

$$+ e^{i\omega t}\rho_0'\rho_0^{-2}\nabla\bar{p} + \rho_0'\rho_0^{-2}\nabla p_0' - \rho_0^{-2}\langle\rho_0'\nabla p_0'\rangle_E. \tag{8}$$

Let $\underset{\sim}{u}_0'$ be the turbulent velocities <u>in the absence of oscillations</u>. The equation for $\underset{\sim}{u}_0'$ is obtained by setting $\bar{\underset{\sim}{u}}, \bar{\rho}$, and $\bar{p}$ equal to zero in Eq. (8). We subtract the equation for $\underset{\sim}{u}_0'$ from Eq. (8), and replace $\underset{\sim}{u}'$ by $\underset{\sim}{u}_0'$ in the terms containing the perturbation $\bar{\underset{\sim}{u}}$. We obtain

$$\frac{\partial \underset{\sim}{u}'}{\partial t} = \frac{\partial \underset{\sim}{u}_0'}{\partial t} - e^{i\omega}\,\bar{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{u}_0' - e^{i\omega t}\underset{\sim}{u}_0'\cdot\nabla\bar{\underset{\sim}{u}} - \underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}' + \underset{\sim}{u}_0'\cdot\nabla\underset{\sim}{u}_0' + \langle\underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}'\rangle$$
$$+ e^{i\omega t}\bar{\rho}\,\rho^{-2}\,\nabla p_0' + e^{i\omega t}\,\rho_0'\,\rho_0^{-2}\,\nabla\bar{p}. \tag{9a}$$

Neglecting quadratic terms in the turbulent quantities, Eq. (9a) reduces to

$$\frac{\partial \underset{\sim}{u}'}{\partial t} = \frac{\partial \underset{\sim}{u}_0'}{\partial t} - e^{i\omega t}\,\bar{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{u}_0' - e^{i\omega t}\,\underset{\sim}{u}_0'\cdot\nabla\bar{\underset{\sim}{u}} + e^{i\omega t}\,\bar{\rho}\rho_0^{-2}\,\nabla p_0'$$
$$+ e^{i\omega t}\,\rho_0'\rho_0^{-2}\nabla\bar{p}. \tag{9b}$$

It is clear that the term $\underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}' - \underset{\sim}{u}_0\cdot\nabla\underset{\sim}{u}'_0 - \langle\underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}'\rangle$ in Eq. (9a) need not be small for fully developed turbulence. This approximation finds justification more readily in the case of random waves (see Moffat 1978, p. 156). It should be stressed, however, that this "first order smoothing" or "quasi-linear" approximation has been used successfully by Krause (1968) in the derivation of the α-term in dynamo theory and lies at the basis of our understanding of the solar cycle. Equation (9b) relates the turbulent velocities in the presence and absence of oscillations and it is, therefore, the central equation of this paper, since it allows us to evaluate $\langle u'\cdot\nabla u'\rangle$ from Eq. (7). To evaluate this term we consider the following very simple model of turbulence: we assume that the turbulent velocities remain relatively unchanged for a turnover time, τ, and then suddenly change to an uncorrelated velocity field. To illustrate the details of the derivation of $\langle u'\cdot\nabla u'\rangle$ it suffices to retain only the first two terms in the right-hand side of Eq. (9b). Therefore,

$$\underset{\sim}{u}'(t) = \underset{\sim}{u}_0'(t) - \int_0^t e^{i\omega t'}\,\bar{\underset{\sim}{u}}\cdot\nabla\underset{\sim}{u}_0'\,dt' \tag{10}$$

The value of the integration constant does not play a role in our calculations, and we have assumed in Eq. (10) that it vanishes. It is clear that $\langle\underset{\sim}{u}'\cdot\nabla\underset{\sim}{u}'\rangle$ contains the term $\langle\underset{\sim}{u}_0'\cdot\nabla\underset{\sim}{u}_0'\rangle$ and two terms linear in $\bar{\underset{\sim}{u}}$ (we neglect quadratic terms in $\bar{\underset{\sim}{u}}$). Now, by definition,

$<\underset{\sim}{u}'_0 \cdot \nabla \underset{\sim}{u}'_0> = e^{i\omega t} \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\omega t} \underset{\sim}{u}'_0 \cdot \nabla \underset{\sim}{u}'_0 \, dt$. This quantity vanishes, since the frequency ω is of no particular significance for the unperturbed turbulence. The two terms linear in $\underset{\sim}{\bar{u}}$ are

$$- < \int_0^t e^{i\omega t'} (\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(t'))dt' \cdot \nabla \underset{\sim}{u}'_0(t)>$$

$$- <(\underset{\sim}{u}'_0(t) \cdot \nabla) \int_0^t e^{i\omega t'} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0 (t')dt'>.$$

We proceed now to derive the first term designated hereafter by $\underset{\sim}{A}$. By definition

$$\underset{\sim}{A} = -\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\omega t} \underset{\sim}{f}(t) \cdot \nabla \underset{\sim}{u}'_0(t)dt \tag{11a}$$

where

$$\underset{\sim}{f}(t) = \int_0^t e^{i\omega t'} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(t')dt'. \tag{11b}$$

Our model of turbulence allows us to write

$$\underset{\sim}{f}(t) = \sum_{n=0}^{N-1} \int_{n\tau}^{(n+1)\tau} e^{i\omega t'} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(t')dt'$$

$$+ \int_{N\tau}^{t} e^{i\omega t'} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(t')dt' = \frac{(e^{i\omega\tau}-1)}{i\omega} \sum_{n=0}^{N-1} e^{i\omega n\tau} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(n)$$

$$+ \frac{e^{i\omega t}-e^{i\omega N\tau}}{i\omega} \underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(N), \tag{12}$$

where $\underset{\sim}{u}'_0(n)$ is the value of the turbulent velocity field in the interval $(n\tau, (n+1)\tau)$ and $N = t/\tau$ with the usual convention used in numerical coding. We can write for $\underset{\sim}{A}$ (the limit $M \to \infty$ is understood)

$$\underset{\sim}{A} = - \frac{1}{M\tau} \sum_{m=0}^{M-1} \int_{m\tau}^{(m+1)\tau} e^{-i\omega t} \underset{\sim}{f}(t) \cdot \nabla \underset{\sim}{u}'_0(t)dt. \tag{13}$$

In the interval $(m\tau,(m+1)\tau)$, $\underset{\sim}{f}(t)$ is given by Eq. (12) with N=m. Substituting Eq. (12) into Eq. (13) we find

$$\underset{\sim}{A} = -\lim_{M\to\infty} [(\frac{e^{i\omega t}-1}{i\omega})^2 \frac{e^{-i\omega t}}{M\tau} \sum_{m=0}^{M-1} \sum_{n=0}^{m-1} e^{i\omega t(n-m)} (\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(n)) \cdot \nabla \underset{\sim}{u}'_0(m)]$$

$$- \lim_{M\to\infty} [\frac{1}{i\omega} \{1 + \frac{1}{i\omega\tau} (e^{-i\omega\tau}-1)\} \frac{1}{M} \sum_{m=0}^{M-1} \{\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(m)\} \cdot \nabla \underset{\sim}{u}'_0(m)]. \tag{14}$$

The first term is a sum over uncorrelated velocity fields $\underset{\sim}{u}'_0(n)$, $\underset{\sim}{u}'_0(m)$

with $n \neq m$ and, therefore, it vanishes. By the ergodic theorem the sum in the second term is equal to $<\{\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0(m)\} \cdot \nabla \underset{\sim}{u}'_0(m)>_E$ where the bracket denotes an ensemble average.

Proceeding in an analogous way with the other terms of $<\underset{\sim}{u}' \cdot \nabla \underset{\sim}{u}'>$ and neglecting the correlations $<\underset{\sim}{u}'_0 \cdot \nabla p'_0>_E$, and $<\underset{\sim}{u}'_0 \, \rho'_0>_E$ we finally obtain

$$<\underset{\sim}{u}' \cdot \nabla \underset{\sim}{u}'> = - \frac{e^{i\omega t}}{i\omega} \left[1 + \frac{1}{i\omega\tau} (e^{-i\omega\tau} - 1)\right] <\underset{\sim}{B}>_E. \tag{15a}$$

$$<\underset{\sim}{B}>_E = <\underset{\sim}{u}'_0 \cdot \nabla(\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0 + \underset{\sim}{u}'_0 \cdot \nabla \underset{\sim}{\bar{u}}) + (\underset{\sim}{\bar{u}} \cdot \nabla \underset{\sim}{u}'_0 + \underset{\sim}{u}'_0 \cdot \nabla \underset{\sim}{\bar{u}}) \cdot \nabla \underset{\sim}{u}'_0>_E. \tag{15b}$$

It is of interest to consider the limit $\omega\tau \to 0$ in Eq. (15). For values of t such that ωt is small, we obtain

$$<\underset{\sim}{u}' \cdot \nabla \underset{\sim}{u}'> = - \frac{\tau}{2} <\underset{\sim}{B}>_E \tag{16}$$

This is essentially Elsässer's (1966) expression for $<\underset{\sim}{u}' \cdot \nabla \underset{\sim}{u}'>$ valid for steady flows. The above formalism is, therefore, generalized in the present paper for the case of oscillating mean flows. Dropping the subscript 'o' in Eq. (15b), we can write for the i-th component of $<\underset{\sim}{B}>_E$:

$$<B_i>_E = <u'_k \, \partial_k(\bar{u}_j \partial_j u'_i + u'_j \partial_j \bar{u}_i) + (\bar{u}_j \partial_j u'_k + u'_j \partial_j \bar{u}_k) \partial_k u'_i>_E =$$

$$<2u'_k(\partial_k \bar{u}_j)(\partial_j u'_i) + \bar{u}_j \, \partial_j(u'_k \, \partial_k u'_i) + u'_k \, \partial_k(u'_j \, \partial_j \bar{u}_i)>_E. \tag{17}$$

In Eq. (17) repeated indices are summed and ∂_j is the derivative with respect to the j-coordinate. We are particularly interested in studying the effect of the convective motions in the outer solar convection zone on non-radial oscillations. Due to symmetry properties, the correlation coefficients for these motions satisfy certain relations as, for example, $<u'_i u'_j>_E = 0$ unless $i = j$; $\partial_j <u'^2_i> = 0$ unless $j = r$. With the help of these relations it is found that

$$<2u'_k(\partial_k \bar{u}_j)(\partial_j u'_i)>_E = \partial_r <u'^2_i>_E \, \partial_i \bar{u}_r, \tag{18a}$$

$$<\bar{u}_j \, \partial_j(u'_k \, \partial_k u'_i)>_E = \frac{1}{2} \bar{u}_r \partial^2_r <u'^2_r>_E, \tag{18b}$$

$$<u'_k \, \partial_k(u'_j \, \partial_j \bar{u}_i)>_E = \frac{1}{2} \partial_r <u'^2_r>_E \partial_r \bar{u}_i + <u'^2_j>_E \partial^2_j \bar{u}_i. \tag{18c}$$

Therefore,

$$<\underset{\sim}{B}>_E = (\partial_r v) \, \nabla \bar{u}_r + v \nabla^2 \underset{\sim}{\bar{u}} + \frac{1}{2} (\partial_r v)(\partial_r \underset{\sim}{\bar{u}}) + \frac{1}{2} \hat{i}_r \, \bar{u}_r \partial^2_r \, v, \tag{19a}$$

where $\hat{i}_r$ is the unit vector in the radial direction and

$$v = \langle u_i'^2 \rangle_E = \frac{1}{3} \langle \underset{\sim}{u}'^2 \rangle_E. \tag{19b}$$

The mean momentum equation (6) can, therefore, be written

$$\frac{\partial \underset{\sim}{U}}{\partial t} - \frac{1 + (e^{-i\omega\tau}-1)/i\omega\tau}{i\omega} \left[(\partial_r v)\ \nabla U_r + v\nabla^2 \underset{\sim}{U} + \frac{1}{2}(\partial_r v)(\partial_r U) + \frac{1}{2}\hat{i}_r\ U_r\ \partial_r^2\ v\right] =$$
$$- \frac{1}{\rho_0} \nabla P + \frac{\Gamma}{\rho_0^2} \nabla p_0 \tag{20a}$$

where $\quad \underset{\sim}{U} = \overline{\underset{\sim}{u}}\ e^{i\omega t};\ P = \overline{p}e^{i\omega t};\ \Gamma = \overline{\rho}e^{i\omega t}$ (20b)

b) <u>The Perturbed Eigenfrequency</u>

Let q be any physical quantity ($\underset{\sim}{u}$, p,....). Then $q = q_0 + Q + q'$ where q_0 is the equilibrium value in the absence of oscillations and turbulence, $Q(= \overline{q}\ e^{i\omega t})$ is the oscillatory component and q' is due to the turbulence (see Eqs. (3)). Clearly,

$$Q = e^{i\omega t} \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\omega t}\ q\ dt. \tag{21}$$

We denote by $\underset{\sim}{\xi}$ the displacement due to the oscillations--defined, therefore, by an equation similar to Eq. (21)--and define

$$\delta Q = Q(\underset{\sim}{r} + \underset{\sim}{\xi}, t) - Q_0(\underset{\sim}{r}, t). \tag{22}$$

Clearly, $\partial\underset{\sim}{\xi}/\partial t = \delta\underset{\sim}{U}$ and, e.g., $\delta\nabla U_r = \nabla\delta U_r$. We apply the δ-operator to Eq. (20a) and set $\omega = \omega_0$ (the unperturbed frequency) in the perturbation terms of Eq. (20a). We obtain

$$-\rho_0\omega^2\underset{\sim}{\xi} - \rho_0(1 + (e^{-i\omega\tau_0} - 1)/i\omega_0\tau)\left[(\partial_r v)\nabla\xi_r + v\nabla^2\underset{\sim}{\xi} + \frac{1}{2}(\partial_r v)(\partial_r\underset{\sim}{\xi})\right.$$

$$\left. + \frac{1}{2}\hat{i}_r\ \xi_r\ \partial_r^2 v\right] = \rho_0\ \delta(-\nabla P/\rho_0 + \Gamma\nabla\rho_0/\rho_0^2). \tag{23}$$

Lynden-Bell and Ostriker (1967) and Schutz (1979) have developed variational principles to determine the eigenfrequencies of an equation like (23). We define

$$a_{\underset{\sim}{\xi}} = \int \rho_0 \underset{\sim}{\xi}^* \underset{\sim}{\xi}\ d^3\underset{\sim}{r}, \tag{24a}$$

$$b_{\underset{\sim}{\xi}} = \int \rho_0 \underset{\sim}{\xi}^*(1 + (e^{-i\omega\tau_0}-1)/i\omega_0\tau)\left[(\partial_r v)\nabla\xi_r + v\nabla^2\underset{\sim}{\xi}\right.$$

$$+ \frac{1}{2}(\partial_r v)(\partial_r \underset{\sim}{\xi}) + \frac{1}{2}\hat{i}_r \xi_r \partial_r^2 v]d^3\underset{\sim}{r}, \tag{24b}$$

$$P_{\underset{\sim}{\xi}} = \int \overset{*}{\underset{\sim}{\xi}} \ \mathbf{P} \cdot \underset{\sim}{\xi}\, d^3\underset{\sim}{r}, \tag{24c}$$

where **P** is given by Eq. (25) of Lynden Bell and Ostriker's paper. Multiplying Eq. (23) by $\underset{\sim}{\xi}^*$ and integrating over the volume, we find

$$-\omega^2 a_{\underset{\sim}{\xi}} - b_{\underset{\sim}{\xi}} + p_{\underset{\sim}{\xi}} = 0. \tag{25}$$

Let

$$\omega = \omega_0 + \omega_1;\ \underset{\sim}{\xi} = \underset{\sim}{\xi}_0 + \underset{\sim}{\xi}_1, \tag{26}$$

where the subscript '$_0$' denotes now the eigenfrequency and eigenfunction in the absence of the perturbation due to the turbulence. Now for the unperturbed state a variational principle holds; i.e., the quantity $-\omega_0^2 a_{\underset{\sim}{\xi}} + p_{\underset{\sim}{\xi}}$ is of second order in the perturbation. Therefore, $-2\omega_0\omega_1 a_{\underset{\sim}{\xi}0} - b_{\underset{\sim}{\xi}0}$ vanishes to first order. That is,

$$\omega_1 = - b_{\underset{\sim}{\xi}_0}/2\omega_0 a_{\underset{\sim}{\xi}0} \tag{27}$$

A word of caution is needed in relation to Eq. (27): It is based on Eq. (19a) for $\langle\underset{\sim}{B}\rangle_E$ and, consequently, in the limit of steady flows the Reynold stresses contain a term, $-\frac{\tau}{2}\frac{1}{2}\hat{i}_r \bar{u}_r \partial_r^2 v$ which does not contain a derivative of the mean flow. The physical significance of this term is doubtful and, again for steady flows, it can be shown that a more satisfactory theory of "turbulent viscosity" does not include this term. Agreement between the results for oscillatory flows (for $\tau\omega \to 0$) and steady flows (cf. paper quoted above) is obtained if the Reynolds stresses are given by $\partial_j(\rho_0 u'^j u'^i)$ with

$$\underset{\sim}{u}'(t) = \underset{\sim}{u}'_0(t) - \int_0^t e^{i\omega t'}\underset{\sim}{u}'_0 \cdot \nabla\bar{\underset{\sim}{u}}\, dt'. \tag{28}$$

It can then be readily shown that $\omega_1 = - b'_{\underset{\sim}{\xi}_0}/2\omega_0 a_{\underset{\sim}{\xi}_0}$ with

$$b_{\underset{\sim}{\xi}'_0} = (1 + (e^{-i\omega_0\tau} - 1)/i\omega_0\tau)\, I,$$

$$I = \int \underset{\sim}{\xi}_0^* \cdot [\partial_r(\rho_0 v)\{\nabla\xi_{0r} + \partial_r\underset{\sim}{\xi}_0\} + \rho_0 v\{\nabla \mathrm{div}\, \underset{\sim}{\xi}_0 + \nabla^2\underset{\sim}{\xi}_0\}]d^3\underset{\sim}{r}; \tag{29}$$

that is,

$$\omega_1/\omega_0 = F(\omega_0, \tau, a_{\underset{\sim}{\xi}_0})I,$$

$$F(\omega_0, \tau, a_{\underset{\sim}{\xi}_0}) = - (1/2\omega_0^2 a_{\underset{\sim}{\xi}_0})[1 - \sin\omega_0\tau/\omega_0\tau + i(1 - \cos\omega_0\tau)/\omega_0\tau]. \tag{30}$$

DISCUSSION

It is clear that Eq. (30) allows for the damping or excitation of the oscillations by the turbulent motions as well as for a frequency shift. We notice that the imaginary term in Eq. (30)--namely, $i(1-\cos\omega_0\tau)/\omega_0\tau$, is an oscillatory function of $\omega_0\tau$. The maximum value occurs at a period of oscillation P, such that $P \sim 8\tau/3$. That is, P and τ are comparable (see Goldreich and Keeley 1977).

To obtain a very rough order-of-magnitude estimate for ω_1, we neglect the last two terms in I and retain only the radial component of $\underset{\sim}{\xi}_0 [= [\eta_{or}, \eta_{oh}\, \partial/\partial\theta, \eta_{oh}(\sin\theta)^{-1}\, \partial/\partial\phi]\, Y_\ell^m(\theta,\phi)]$. We obtain

$$\omega_1/\omega_0 = F(\omega_0, \tau, a_{\xi_0}) \int \partial_r(\rho_0 v)\, \partial_r(\eta_{or}^2)\, d^3\underset{\sim}{r}. \tag{31}$$

In Table 1, $\rho_0 v$ $(=1/3\rho_0\langle\underset{\sim}{u}'^2\rangle)$ is tabulated for surface values of the solar convection zone.

Table 1. Values of the kinetic energy of the turbulent convective motions as a function of depth in the solar convection zone

d(km)	50	100	200	300	400	500	600
$\rho_0 v/10^3$(cgs)	1.2	7.4	6.4	6.3	6.6	7.1	7.6
d	700	800	900	10^3	10^4	10^5	
$\rho_0 v/10^3$	8.2	8.7	9.3	10.1	$1.3\,10^2$		$6.8\,10^2$

It is clear from Eq. (31) that we need to calculate the following expression:

$$A = \int\partial_r(\rho_0 v)\partial_r(\eta_{or}^2)d^3\underset{\sim}{r}/(2\int\rho_0\eta_{or}^2\, d^3\underset{\sim}{r}). \tag{32}$$

With the exception of the surface layers of the convection zone, Table 1 shows that $\partial_r(\rho_0 v) \sim 10^{-4}$(cgs). Figure (9) of Ulrich and Rhodes (1983) suggests that the following relations have some validity:

$\rho_0(S)\eta_{or}^2(S) \sim 10\, \langle\rho_0\eta_{or}^2\rangle$ where S and the bracket denote a surface and the average value, respectively. Keeping in mind that η_{or}^2 decreases rapidly with depth, we find $A \sim 10^{-4}[\eta_{or}^2(S)\rho_0(S)/\rho_0(S)]/2d\langle\rho_0\eta_{or}^2\rangle$ where d is the depth of the convection zone. With $\rho_0(S) = 3\; 10^{-6}$ g cm^{-3} (density at a depth of 1000 km), $A \sim 10^{-8}$ s^{-2} and $\omega_1/\omega_0 \sim 2.5\; 10^{-5}$ for oscillations with a period of 5 minutes. It should finally be noticed that since $\partial_r(\rho_0 v) > 0$ (a thin surface layer excepted) Equation (31) shows that the turbulent motions will act as a driving term for the oscillations if $\partial_r(\eta_{or}^2) > 0$ in the region contributing predominantly to the integral in Eq. (31).

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