Astronomy 82 - Problem Set #4

Due: Friday, May 9, 2008 in class or to Ian by noon.

Reading: Chapter on The Milky Way

Problems:

1) **The Crab pulsar radiates 1 x 10^{31} Watts when integrated over all wavelengths. If this is the primary energy loss for the pulsar and it primarily comes from a loss of rotational energy, then at what rate is it slowing down its rotation rate? Assume that its current period is 0.033 s, M = 1.4 M_{sun}, and R = 1.1x 10^4 m. Express your answer as dP/dt (period change as a function of time).**

The pulsar is radiating energy (which we observe as radio waves). Since the total energy in the universe must be conserved, this radio energy must come from somewhere. In this case, it is taken out of the rotational kinetic energy of the pulsar: thus, it gradually slows down.

We're interested in a relation between the pulsar luminosity and its rotational period. In general, the kinetic energy of a rotating body is given by $E=(1/2)\,I\,\omega^2$. Since we want this in terms of the rotational period, we can convert from omega to period:

$E=\frac{1}{2}I\left(\frac{2\pi}{P}\right)^2=2\pi^2 I\,P^{-2}$. Since Luminosity is the time derivative of energy, we are now in a position to relate the quantities we are interested in:

$L=\frac{dE}{dt}=\frac{d}{dt}\left(2\pi^2 I\,P^{-2}\right)=2\pi^2 I(-2)\,P^{-3}\frac{dP}{dt}$, or:

$L=-4\pi^2 I\,P^{-3}\frac{dP}{dt}$. Rearranging this in terms of the quantity we want – the rate of change of the period – gives:

$$\frac{dP}{dt}=\frac{-L\,P^3}{4\pi^2 I}$$

If we assume that this neutron star is a homogeneous sphere (not really true, but a simple approximation), then its moment of inertia is just:

$I_{\text{sphere}}=\frac{2}{5}M\,R^2$, and so the final rate of change of period we get is:

$$\frac{dP}{dt}=\frac{-L\,P^3}{4\pi^2}\frac{5}{2\,M\,R^2}=-\frac{5}{8\pi^2}\frac{L\,P^3}{M\,R^2}$$

Sticking in numbers, the answer is:

$\frac{dP}{dt}=-\frac{5}{8\pi^2}\frac{L\,P^3}{M\,R^2}=6.72\times10^{-14}\,s/s=7\times10^{-14}\,s/s=2\,\mu\text{s/yr}$ This is (approximately) the true value (http://astronomy.swin.edu.au/cosmos/P/Pulsar+Characteristic+Age)

2) **The mass of a pulsar is 1.5 M_{sun}, radius 10 km, and rotation period 0.033 s. What is the angular momentum of the pulsar? Variations of 0.0003 s are observed in the period. If they are due to radial oscillations ("starquakes"), how large in radius are these oscillations?**

The key to this problem, as I said in discussion section, is that it focuses on the conservation of angular momentum – this is further evidenced by the fact that you are explicitly asked to calculate the amount of angular momentum in the system.

It's fairly straightforward to calculate the angular momentum: $L_{ang} = I\omega$. Again, we'll make the (standard) simplifying assumption that the neutron star's moment of inertia is just $I_{\text{sphere}} = \frac{2}{5} M R^2$, and calculate angular velocity with respect to period in the usual way. Then, we have:

$$L_{ang} = \frac{2}{5} M R^2 \frac{2\pi}{P} = \frac{4\pi}{5} \frac{M R^2}{P} = 2.3 \times 10^{40} \text{kg m}^2 \text{s}^{-1}$$

To find out how much the star might be varying in radius during these "starquakes" (or "glitches," as they are also known), we want to find the dependence of radius on rotational period. We have the relation above that:

$L_{ang} = \frac{4\pi}{5} \frac{M R^2}{P}$, which we can rearrange to find the dependence of R on P:

$R = \left(\frac{5 L_{ang}}{4\pi M} \right)^{1/2} P^{1/2}$. Now, we take the derivative of R with respect to P in order to find out how R varies when P changes:

$\frac{dR}{dP} = \left(\frac{5 L_{ang}}{4\pi M} \right)^{1/2} \left(\frac{1}{2} \right) P^{-1/2} = \frac{R}{2P}$ (note that with this substitution, we don't have to worry about calculating that mess of constants out in front!)

So that for a given change of period, the fractional change in radius is twice the fractional change in period:

$$\frac{dR}{R} = \frac{1}{2} \frac{dP}{P}.$$

The change in radius we expect to see from these changes in rotational period are then given by:

$$dR = \frac{R}{2} \frac{dP}{P} = \frac{10^4 \text{m}}{2} \frac{0.3 \mu\text{s}}{33 \mu\text{s}} = 45 \text{m}$$

(you might even say just 50 m, given the low level of precision provided in the period change.)

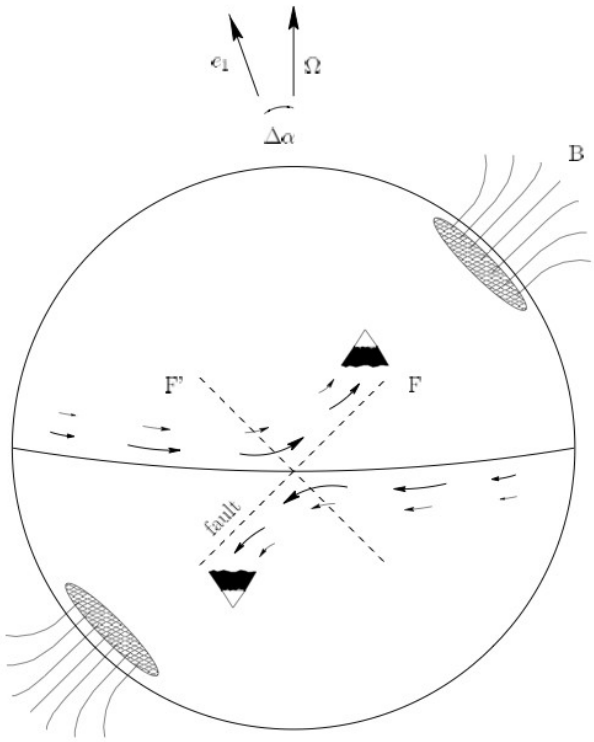

FIG. 6.—Starquake. In the absence of a magnetic field, the material is equally likely to begin breaking along faults F and F' anywhere on the equator. These fault planes are perpendicular to the stellar surface. In the presence of magnetic stresses, faulting along F, which is far from the magnetic poles and directed toward them. A starquake along F creates "mountains" (indicated by the snowcapped peaks) and shifting the largest principal axis of inertia to a new direction e_1 (fixed in the star).

Diagram of a model of neutron "starquakes" at left taken from Franco, Link, & Epstein (2000): available online on Astro-ph at http://arxiv.org/abs/astro-ph/9911105

This diagram at right shows a measurement of the pulse profile from a pulsar. Because of the interstellar medium, different-frequency radio waves travel at different speeds (just like a prism dispersing visible light) – thus measurements are made at many frequencies, shifted, and co-added. This is taken from Wang et al (2001), which also discusses observations of a starquake in the Crab Pulsar: http://arxiv.org/abs/astro-ph/0111006

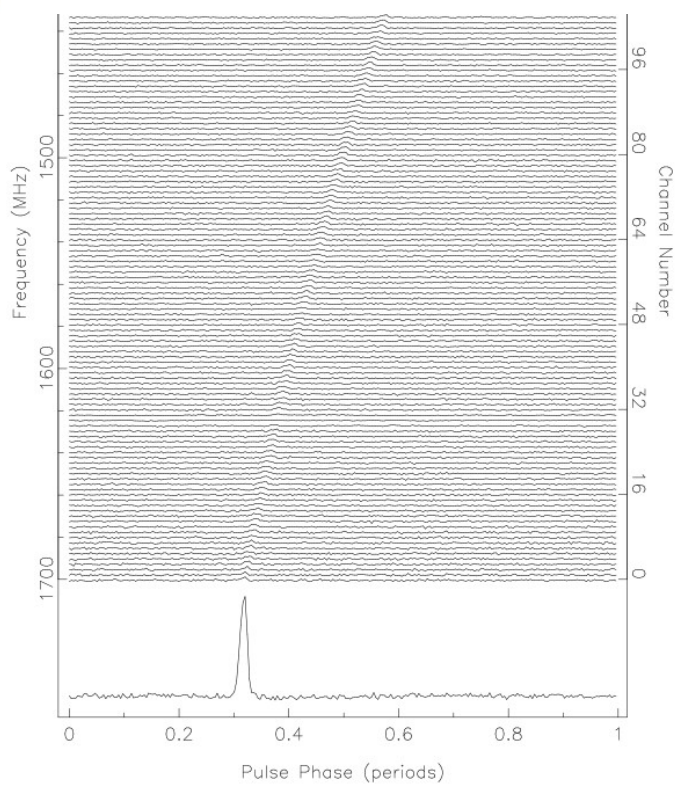

Figure 2. Pulse profiles of PSR B1933+16. The upper part shows the profile in each of the 128 2.5MHz channels, the lower part is the sum of all the channels after appropriate shifts for dispersion have been applied.

3) **In "Dragon's Egg" by Robert L. Forward a spaceship orbits a neutron star at a distance of 406 km from the center of the star. The orbital period is the same as the rotation period of the star, 0.1993 s.**

a) **Find the mass of the star and the gravitational acceleration felt by the spaceship.**

This part of the question asks about some standard features of orbital mechanics for a spacecraft in an "astrostationary orbit". One object orbiting a more massive object; trying to find the mass of that object; this should remind you of the concept of "enclosed mass" in Kepler's Laws. This can be expressed in terms of solar units as:

$(m_1+m_2)(P/1\text{ yr})^2=(a/1\text{ AU})^3$. Assuming the beings in the spacecraft aren't flying entire stars around, their ship probably weighs much less than a neutron star. Therefore, the mass of the star is given by the following relation:

$$m_{ns}=(a/1\text{ AU})^3(P/1\text{ yr})^{-2}$$

$$m_{ns}=\left(\frac{4.06\times10^5\text{ m}}{1.50\times10^{11}\text{ m}}\right)^3\left(\frac{0.1993\text{ s}}{3.16\times10^7\text{ s}}\right)^{-2}=0.498\,M_{\text{sun}}$$

You can find the gravitational acceleration in either of two ways, since the centripetal acceleration can be described by either of the following two relations:

$g=\frac{G\,m_{ns}}{a^2}$ or $a_c=\frac{v^2}{a}$. I'll use the former, and so I find that there's an acceleration:

$$g=\frac{G\,m_{ns}}{a^2}=\frac{6.67\times10^{-11}\times9.92\times10^{29}\text{ kg}}{(4.06\times10^5\text{ m})^2}=4.01\times10^8\text{ m s}^{-2}=4.09\times10^7\,g_{\text{earth}}$$

b) **What is the effect of the gravitation on a 175 cm tall astronaut, if she stands with her feet pointing towards the star? And if she is lying tangential to the orbit?**

What tears a things apart near a neutron star or black hole? The same thing that causes the tides on earth: the varying effect of gravity over position. We want to know how much gravitational acceleration will change with respect to position, so we take the derivative with respect to distance:

$\frac{dg}{dr}=\frac{-2G\,m_{ns}}{r^3}=2\frac{g}{r}$. This means that over a distance "dr", the gravitational acceleration "g" changes by an amount "dg":

$dg=-\frac{2\,g\,dr}{r}$. If she is perpendicular to the orbit, her head and toes are separated by dr = 1.75 m... if lying tangential, her chest and back are separated by perhaps only dr = 10 cm. This results in gravitational stresses on her body of 3460 g_{earth} and ~ 200 g_{earth}, respectively.

4) **A photon leaves the surface of a star at a frequency ν_e. An very distant observer finds that its frequency is ν. If the difference is due to gravitation only, then the change in the energy of the photon, h($d\nu$) equals the change in its potential energy. Find the relation between ν and ν_e assuming the mass and radius of the star are M and R. How much will solar radiation redshift on its way to the Earth?**

To derive the low-order approximation of gravitational redshift, follow these steps: For this discussion we'll give a fictional mass "m" to the photon such that its energy is mc^2. Initially, its energy is:

$$h\nu_1 = mc^2 - \frac{GMm}{R}$$

After escaping and traveling far, far away from any other masses, its energy will simply be:

$$h\nu_2 = mc^2$$

The change in the frequency of the light can then be written as:

$$\frac{\nu_1}{\nu_2} = \frac{mc^2 - \frac{GMm}{R}}{mc^2} = 1 - \frac{GM}{Rc^2}$$ (which is only a low-order approximation to the equation given in your textbook on page 444, Eq. B.3).

Therefore the fractional change in frequency will be:

$$\frac{\Delta\nu}{\nu_2} = \frac{\nu_2 - \nu_1}{\nu_2} = 1 - \frac{\nu_1}{\nu_2} = 1 - \left(1 - \frac{GM}{Rc^2}\right) = \frac{GM}{Rc^2}$$, and the change in wavelength for a typical yellow, 550 nm photon will be:

$$\Delta\lambda = \frac{GM}{Rc^2}\lambda_2 = \frac{(6.67\times10^{-11})\times(2\times10^{30}\,\text{kg})}{(7\times10^{8}\,\text{m})\times(3\times10^{8}\,\text{m s}^{-1})^2}\times(5.5\times10^{-7}\,\text{m}) = 1.2\times10^{-12}\,\text{m}$$

That is, the fractional change in wavelength (or frequency) will be approximately 2 x 10^{-6}.