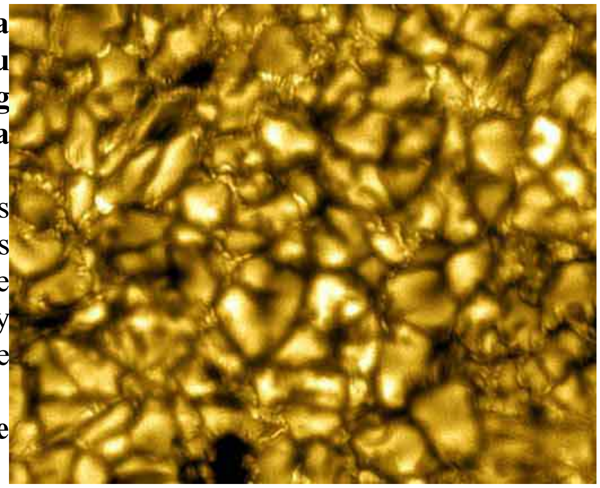
Astronomy 82 - Problem Set #2 Solutions (IJC)

Due: Friday, April 18, 2008 before noon.

Problems:

1) **Assume that a solar granule has a mean molecular weight of 1.34 amu (atomic mass units), a density of 2 kg m^{-3}, and is roughly spherical with a diameter of 1000km.**



"Granules," "convective cells," and "gas parcels" are all roughly synonymous terms used in the description of the qualitative Mixing Length Theory (MLT), which describes convective processes.

a) **What is the total mass of the granule?**

Mass is density times volume. We are given the density, and we're quite familiar with the volume of a sphere. Paying attention to units, we have:

$$m=\rho V=(2\,kg\,m^{-3})(\frac{4}{3}\pi(5\times10^{5}\,\mathrm{m})^{3})=1.04\times10^{18}\,kg=10^{18}\,\mathrm{kg}$$

b) **If its average temperature is 7000 K, what is the amount of thermal energy within the granule? Assume that the average atom has an energy of (3/2)kT.**

The total energy is the amount of energy per particle times the total number of particles. We know that there are an Avogadro's Number of these particles (mostly atoms and ions) in 1.34 grams, so the total number of particles in the sun is:

$$N=\left(\frac{m_{\mathrm{parcel}}}{1.34\,g}\right)N_{A}=\left(\frac{10^{18}\,\mathrm{kg}}{1.34\times10^{-3}\,\mathrm{kg}}\right)\times(6.0\times10^{23})=4.5\times10^{44}\,\mathrm{particles}$$

This means that the total thermal energy in the granule is roughly:

$$T=\frac{3}{2}N\,k\,T=1.5\times(4.5\times10^{44})\times\left(\frac{1}{7}\times10^{-22}\,\mathrm{J/K}\right)\times(7000\,\mathrm{K})=6.72\times10^{25}\,\mathrm{J}=7\times10^{25}\,\mathrm{J}$$

Using the exact value of k would give 6.5 x 10^{25} J – but since we don't *really* know the shape of the parcel in any case, we can't really claim to know the thermal energy to that level of precision. It's a very rough calculation.

2) **If the granule is question 1 stays at the top of the photosphere for 10 minutes,**

a) **how much energy does it radiate away if its effective surface temperature is 6000 K, and its emissivity is 1?**

We'll assume the granule radiates as a blackbody and use the Stefan-Boltzmann law, then get total energy radiated via "energy = power times time." The integrated flux emitted by a greybody is given by the relation $F=\epsilon\sigma_{SB}T^{4}$. The total emitted luminosity of the greybody is then the flux times the emitting surface area: $L = F\, A = \epsilon \sigma_{SB} A T^4$. As the image of solar granules above demonstrates, these granules are not strictly spherical – nonetheless, we will approximate them as hemispheres protruding from the surface of the sun – thus the effective surface area will be $A = 2\pi r^2$ (anything within a factor of several is fine). So the power emitted is:

$$L = 4\pi d_1^2 F_1 = 4\pi d_2^2 F_2$$

$$L = 6.28 \times (5\times 10^5\,\mathrm{m})^2 \times 1 \times (5.67\times 10^{-8}) \times (6000\,K)^4 = 1.15\times 10^{20}\,W$$

Therefore the total radiated energy is:

$$P = L\,t = (1.15\times 10^{20}\,\mathrm{W}) \times \left(10\,\mathrm{min} \times \frac{60\,\mathrm{s}}{1\,\mathrm{min}}\right) = 6.9\times 10^{22}\,\mathrm{J} = 7\times 10^{22}\,\mathrm{J}$$

Based on the uncertainty in the surface area, anything within a factor of a few is fine.

b) By how much does its temperature drop during the 10 minutes?

The simplified way to do this is to just look at the fractional amount of energy that was radiated from the granule. Since only 0.1 % of the total thermal energy was radiated away, the average temperature will also only drop by 0.1 %. Since it started at 7000 K, the temperature will drop about 7 K. Not much, but enough for the granule to lose its buoyancy and sink back below the photosphere.

3) How long does it take for the sun to convert one Earth mass of hydrogen into helium?

First, an initial check: what's a *reasonable answer*? The sun weighs about 300,000 as much as the earth, so if the time were only "1 hour" then the total lifetime of the sun would be less than 300,000 hours (about 34 years). So it's more likely to be on the order of millions of hours (billions of seconds) – or even greater.

We'll use the fundamental relation that "fuel energy" = "radiated power" times "emitting time," where the "fuel" in this case is the sun's hydrogen. How much energy is released by fusing hydrogen? There are several ways to find this, but a handy quantity to remember is listed on your OoMA sheet: "**Fusing H to He yields 0.7% of mc^2**." Thus fusing four protons to form one alpha particle (a Helium nucleus) yields (0.007 x 4 x 938 MeV) = 26 MeV of liberated energy.

In this case, the total liberated energy by fusing one earth mass of hydrogen will be:

$$E_f = 0.007\, m_E c^2 = 0.007 (6\times 10^{24}\,\mathrm{kg})(3\times 10^8\,\mathrm{m/s})^2 = 3.8\times 10^{39}\,\mathrm{J}$$

(compare this to the energy radiated by a granule of solar material, above!)

Since the sun's energy output is essentially constant (a solar luminosity), then the time necessary to release this much energy is given by:

$$t = \frac{E_f}{L_{\mathrm{sun}}} = \frac{3.8\times 10^{39}\,\mathrm{J}}{4\times 10^{26}\,\mathrm{W}} = 9.5\times 10^{12}\,\mathrm{s} = 3\times 10^5\,\mathrm{yr}$$

Significantly more than "an hour or so," as discussed initially above – and comfortably less than the sun's total lifetime.

4) **What is the escape velocity from the surface of the sun? What temperature does H gas need to be to achieve this as an average speed?**
You should have already done these last week, and the solutions were posted online.

5) **The solar constant, i.e., the flux density of the solar radiation at the distance of the Earth is 1390 W m^{-2}.**

a) **Find the flux density on the surface of the Sun, when the apparent diameter is 32 arcminutes.**
There are more complicated ways to do this, but the most straightforward is to remember that flux density obeys an inverse square law with distance. The relevant distances here are the distance from the effective "center of radiation" (the center of the sun) from: (1) the Earth (1 AU) and (2) the sun's surface (i.e., the radius of the sun).

You can look up the radius of the sun (700 000 km), but it's also useful to see that you can calculate it from its distance and apparent size: $\theta_D = 2 r_{\text{sun}} / d_{\text{sun}}$. You can rearrange this (and convert from arcminutes to radians) to find the sun's radius:

$$r_{\text{sun}} = d_{\text{sun}} \theta_D / 2 = 0.5 (1\,\text{AU}) (32\,\text{arcmin} \times \frac{2\pi\,\text{rad}}{360 \times 60\,\text{arcmin}})$$

$$r_{\text{sun}} = 4.7 \times 10^{-3}\,\text{AU} = 7.0 \times 10^{8}\,\text{m} \,.$$

Now, we apply the inverse square law. It derives from the constancy of luminosity: $L = 4\pi d_1^2 F_1 = 4\pi d_2^2 F_2$, and so $F_1 d_1^2 = F_2 d_2^2$. Let "case 1" be at the earth's orbit and "case 2" be on the sun's surface. Then,

$$F_{\text{sun}} = F_{\text{earth}} \left(\frac{d_{\text{earth}}}{d_{\text{sun}}} \right)^2 = (1390\,\text{W m}^{-2}) \left(\frac{1\,\text{AU}}{4.7 \times 10^{-3}\,\text{AU}} \right)^2 = 6.4 \times 10^{7}\,\text{W m}^{-2}$$

b) **How many square meters of solar surface is needed to produce 1000 megawatts?**
A little confusing, since the sun's surface doesn't "produce" energy ... it merely radiates it away after the energy has made its way all the way through the sun's structure and out into the photosphere. Nonetheless, just look at the units of power and of flux to figure out the relation to surface area: P = F A, so A = P / F. Thus,

$$A = P / F = \frac{10^{9}\,\text{W}}{6.4 \times 10^{7}\,\text{W m}^{-2}} = 15.6\,\text{m}^2 = 16\,\text{m}^2$$

6) **If 4.5 billion years ago the sun had a surface temperature of 5000 K and a radius of 1.02 modern solar radii, what was the solar constant at 1 AU?**
First we'll see how much the solar luminosity has changed in the last 4.5 Gyr, and then we'll use that to determine how the solar constant has changed. Apparently the sun has shrunk slightly and increased in temperature since then... we'll have to see whether the decrease in radius (decreasing emitted power) or the increase in temperature (increasing emitted power) will have a greater effect.

As discussed in problem 2, the luminosity of a radiating object is $L=4\pi R^2\sigma_{SB}T^4$ (assuming the object is a blackbody – a good approximation for the sun). Thus as the radius and effective temperature evolve over time, we see that the luminosity will change as:

$$\frac{L_1}{L_2}=\left(\frac{R_1}{R_2}\right)^2\left(\frac{T_1}{T_2}\right)^4 .$$

The luminosity of the sun earlier in it's lifetime, then, should be:

$$L_1=\left(\frac{R_1}{R_{\text{now}}}\right)^2\left(\frac{T_1}{T_{\text{now}}}\right)^4 L_{\text{now}}=\left(\frac{1.02}{1}\right)^2\left(\frac{5000}{5700}\right)^4 L_{\text{now}}=0.62\,L_{\text{sun}}$$

Out at 1 AU, the only varying factor we are considering is the fact that the sun changed in total luminosity. Thus the flux 1 AU from the sun will decrease by the same amount as the luminosity, and the solar constant then would have been:

$$F_1=0.62\,F_{\text{now}}=0.62\times1390\text{ W m}^{-2}=860\text{ W m}^{-2}$$

Of course, significantly less flux would have reach the current position of the earth's orbit – there would have been substantially more intervening gas and dust in the formative stages of the solar system's evolution. But this is a nice, simplified answer.

7) Calculate the lifespan for the two stars below. Assume that essentially all of their luminosity comes from the fusion of H into He. Also assume that they begin as 75% H gas (by mass) and that they will die when they have fused 10% of this gas into He.

So, 75% of a star's mass is Hydrogen and about 10% of these atoms will fuse into Helium – thus, 0.075 M_* will undergo fusion. Referring again to your OoMA sheet, you can see that fusing H --> He releases 0.7% mc^2. Therefore, the total energy liberated in a star's lifetime is roughly:

$$E_f=0.007\times0.75\times0.1\times M_*c^2$$

Since the luminosity of a star is equal to its energy output divided by the time it gives off energy, the lifetime of the star should just be $t=E/L$.

a) **High mass star with M = 2 x 10^{32} kg and luminosity L=4 x 10^{32} W.**
A 100 solar mass star, with a luminosity of one million solar luminosities:

$$t=\frac{E}{L}=\frac{0.007\times0.75\times0.1\times(2\times10^{32}\text{ kg})\times(3\times10^8\text{ m/s})^2}{4\times10^{32}\text{ W}}=2.36\times10^{13}\text{ s}=7.5\times10^5\text{ yr}$$

Such massive stars live very briefly; thus we see very few of them.

b) **Low mass star with M = 10^{30} kg and luminosity L=4 x 10^{25} W.**
Only half a solar mass, and a tenth of a solar luminosity:

$$t=\frac{E}{L}=\frac{0.007\times0.75\times0.1\times(10^{30}\text{ kg})\times(3\times10^8\text{ m/s})^2}{4\times10^{25}\text{ W}}=1.18\times10^{18}\text{ s}=3.7\times10^{10}\text{ yr}$$

These stars can last tens of billions of years -- barring further interactions, they will still be burning far into the future of the universe.