

# Astronomy 82 - Problem Set #1 Solutions (IJC)

Due: Friday, April 11, 2008 before noon.

Problems:

- 1) **If a solar mass star is observed to have a periodic velocity shift of 20 m/sec with a period of 3.5 years, what is the mean orbital radius of the companion and what is the projected mass of the companion ( $m \sin(i)$ )? Is this object a planet or a brown dwarf?**

From Keplerian orbital mechanics, the orbital semimajor axes ( $a$ ) and velocities ( $v$ ) of the star and companion depend only on their masses:  $m_* a_* = m_c a_c$  and  $m_* v_* = m_c v_c$ . When making radial velocity measurements we can only measure the component of velocity along our line-of-sight. This results in a relation between the projected velocities,  $m_* v_* \sin i = m_c v_c \sin i$ . However, we actually only measure ( $v_* \sin i$ ) from the Earth.

We therefore have seven quantities -- two masses, orbital radii, and velocities plus the inclination angle -- related by two equations (listed above). We are given  $v_*$  and  $m_*$  and are told we do not need to resolve the ambiguity in "sin  $i$ ." so we have seven degrees of freedom (the quantities) and currently only five constraints (the equations and given, or "ignored," quantities). We need two additional constraints to solve this system -- constraints that relate the desired quantities. As is often the case with orbital systems, we turn to Kepler's laws and basic Newtonian mechanics.

The first equation we will use is Kepler's Third Law. Assuming this companion is much less massive than its host star, we can use it in the form of  $(m_*/M_{\text{sun}}) \times (P/1 \text{ yr})^2 = (a_c/1 \text{ AU})^3$  to find the orbital radius of the companion:  $(a_c/1 \text{ AU}) = (P/1 \text{ yr})^{2/3} \times (m_*/M_{\text{sun}})^{1/3}$ . Thus, the companion's mean orbital radius (really its semimajor axis) is **2.3 AU**.

Next, use the equation of centripetal gravitational force:  $\frac{m_c v_c^2}{a_c} = \frac{G m_* m_c}{a_c^2}$ , which can be simplified and written as  $v_c = \sqrt{\frac{G m_*}{a_c}}$ . Since we already found  $a_c$ , this immediately gives the planet's orbital velocity amplitude: 20 km/s.

We can now return to our initial equation to solve for the final quantity we desire. Plugging in values, we see that  $m_c = \frac{v_*}{v_c} m_* = \frac{1}{1000} m_* = 0.001 M_{\text{sun}} \approx 1 M_{\text{Jupiter}}$ .

A quick look at the [Exoplanet Encyclopedia](#) shows that this planet does not actually exist -- Professor Larkin made it up.

**2) What is the escape velocity from the surface of the sun? What temperature does H gas need to be to achieve this as an average speed?**

To calculate escape velocity, use energy conservation: when in a gravitational potential, what additional kinetic energy is needed to make the system unbound (that is, for the energy to be at least zero)? If the initial energy is less than zero the system will remain bound and the object trying to escape will eventually come hurtling back down – like a baseball hit high into the air. So we write:

$E_i = T + U \geq 0$ , which means that:

$$\frac{1}{2} m v^2 - G m_* \frac{m}{r_*} \geq 0.$$

When this inequality is *just* satisfied the velocity  $v$  is the escape

velocity,  $v_{esc}$ :

$$\frac{1}{2} v_{esc}^2 = \frac{G m_*}{r_*},$$

giving the formula for escape velocity:  $v_{esc} = \sqrt{\frac{2 G m_*}{r_*}}$ . Plugging in

approximate (but easy to remember!) values for these quantities gives:

$$v_{esc} = \sqrt{\frac{2 G m_*}{r_*}} = \sqrt{\frac{2 * (2/3 * 10^{-10}) * (2 * 10^{30})}{7 * 10^8}} = \sqrt{\frac{8}{21} * 10^{12} \text{ m}^2 \text{ s}^{-2}} = 600 \text{ km s}^{-1}.$$

The exact answer is about 618 km/s.

From Kinetic Molecular Theory (and statistical mechanics), the average (really Root-Mean-Square) kinetic energy of a gas particle is given by  $\overline{KE} = \frac{1}{2} m \overline{v}^2 = \frac{3}{2} k_B T$ , where  $k_B$  is the Boltzmann constant. This means the average velocity in terms of gas temperature is given by  $\overline{v} = \sqrt{3 k_B T / m}$ . We want a temperature based on a mean velocity, though, so rearranging this gives

$T = \frac{m \overline{v}^2}{3 k_B}$ . Using the solar escape velocity calculated above, the temperature needed for most hydrogen atoms' velocities to exceed the escape velocity is:

$$T = \frac{m \overline{v}^2}{3 k_B} = \frac{(1.67 * 10^{-27} \text{ kg}) * (6.2 * 10^5 \text{ m/s})^2}{3 * (1.38 * 10^{-23})} = 1.6 * 10^7 \text{ K}.$$

Since the surface of the sun is actually only ~ 5800 K you can see that hydrogen atoms are very tightly gravitationally bound to the sun's surface – hardly any will escape the sun. However, temperatures in the solar corona reach several millions of degrees Kelvin, and a small but non-negligible fraction of particles can escape from the sun there. These escaping particles are the solar wind!

- 3) What is the escape velocity from the top of Jupiter's atmosphere? If the temperature there is about 200 K, what is the rms velocity of helium? Comment on Jupiter's ability to hold on to helium.

Using the formula for calculating escape velocity from the previous problem, we have:

$$v_{esc} = \sqrt{\frac{2 G m_J}{r_J}} = \sqrt{\frac{2 * (2/3 \times 10^{-10}) \times (2 \times 10^{27})}{7 \times 10^7}} = \sqrt{\frac{8}{21} \times 10^{10} \text{ m}^2 \text{ s}^{-2}} = 60 \text{ km s}^{-1}.$$

For the RMS helium gas velocity we can also use our result from above. The mass of helium is approximately four times the mass of a proton, and so we have:

$$\bar{v} = \sqrt{\frac{3 k_B T}{m_{\text{He}}}} = \sqrt{\frac{3 \times (1.38 \times 10^{-23}) \times (200 \text{ K})}{4 \times (1.67 \times 10^{-27} \text{ kg})}} = \sqrt{1.24 \times 10^6 \text{ m}^2 \text{ s}^{-2}} = 1.1 \text{ km/s}.$$

Given the low average speed of helium atoms in Jupiter's atmosphere, there should be relatively little "leakage" of helium from Jupiter out into space. Contrast this with the Earth: its escape velocity is much lower (~11 km/s) and its temperature is a bit higher, so there is a slow, but substantial, "leak" of helium from Earth's atmosphere. The helium we see is being constantly replenished by the radioactive alpha-decay of heavier elements!

- 4) How far from the Sun's center is the common center of mass of Jupiter and the Sun? This is roughly the size of the wobble of the Sun as it moves around the Galaxy. If you were trying to observe this from a distance of 10 pc, what would the angular diameter of the wobble be? If Jupiter was only 1 AU from the Sun, what would the angular wobble be?

From the definition of Center of Mass, the separation  $d$  of two objects from their common center of mass is given by:  $m_* d_* = m_J d_J$ . For Jupiter, the sum of these distances is approximately its semimajor axis  $a$ , about 5.2 AU or 778 million km. The separation of the sun from this barycenter is therefore:

$$m_* d_* = m_J (a - d_*)$$

$$d_* = \frac{a m_J}{m_* + m_J} \approx \frac{m_J}{m_*} a \approx \frac{2 \times 10^{27} \text{ kg}}{2 \times 10^{30} \text{ kg}} \times 5.2 \text{ AU} = 0.0052 \text{ AU} \text{ -- smaller than the radius of the sun!}$$

Using more exact numbers, you get a value of 5 mAU = 740 000 km.

The angular size of this motion from ten parsecs away can be computed from the simple triangle shown here. Although as drawn the relationship is technically

$$\tan \theta = \frac{0.0052 \text{ AU}}{2 \times 10^6 \text{ AU}}, \quad \text{the } \img alt="A right-angled triangle with a very small angle theta at the vertex. The vertical side is labeled 0.0052 AU and the horizontal side is labeled 2 x 10^6 AU." data-bbox="370 745 830 780"/>$$

angle is so small that one can use the "small-angle approximation:"

$$\theta \approx \frac{0.0052 \text{ AU}}{2 \times 10^6 \text{ AU}} = 2.6 \times 10^{-9} \text{ radian} \times \frac{360 \times 60 \times 60 \times 1000 \text{ micro arcsec}}{2 \pi \text{ radian}} = 520 \text{ micro arcsec}$$

This is extremely difficult to measure from the ground – and all but impossible for all but the brightest stars. However, NASA's SIM: PlanetQuest and ESA's GAIA missions will

both be able to perform measurements over 10-100 times more precise than this, possibly discovering earth-mass planets around some of the nearest stars!

If Jupiter were only 1 AU from the sun instead of 5.2 AU away, the sun would be 5.2 times closer to its common barycenter with Jupiter – therefore the astrometric “wobble” would be 5.2 times less: 100 micro arcseconds. Both SIM and GAIA will still easily be able to measure this motion around many stars.

